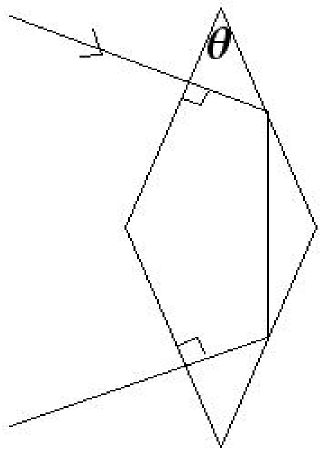
Physics 424H – Modern Optics
Midterm: Tuesday 27th February. 2 hours.
Allowed: 3 hours. Calculator, up to 2 sides of 8 ½×11" paper (equations only, no text).
Answer all questions. Show your working!

1. a) A "Mooney rhomb" is a prism with a parallelogram cross-section (shown below) and used to modify the polarization state of incident light through two total internal reflections.

i) What single value of the apex angle, θ, is required for the light to follow the geometrical path shown?

ii) What is the minimum value of the refractive index of the rhomb such that light entering from vacuum follows this path?



b) Write down mathematical expressions for the electric fields, $\mathbf{E}(\mathbf{r},t)$, of oppositely circularly-polarized plane waves along the z-axis in terms of two waves of the same k linearly polarized along $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$. Define one of them to be "right" and the other "left". Show how linearly polarized light at an arbitrary angle can result from the superposition of these two waves.

c) Consider the propagation of an electromagnetic wave through a medium whose index of refraction depends upon the state of polarization, such that

$$n_{\pm} = \alpha \pm \eta$$

where α and η are real and positive, and the plus and minus signs refer to "right" and "left" circularly-polarized plane waves respectively. By decomposing a linearly-polarized plane wave into two circularly polarized components with the appropriate k values, show that such a wave incident on the medium at $z = 0$ has its plane of polarization rotated as it travels through the medium and find the first value of z at which the plane of polarization is (i) linear again, and (ii) returns to the same plane of polarization at which it entered the medium.

2. A plane monochromatic electromagnetic wave is traveling in *vacuo* with electric field and magnetic fields:

$$\mathbf{E}_{\mathbf{I}}(z,t) = E_{0I}\hat{\mathbf{x}}\cos(kz-\omega t)$$
$$\mathbf{B}_{\mathbf{I}}(z,t) = B_{0I}\hat{\mathbf{y}}\cos(kz-\omega t) \qquad \text{(Eq. 1)}$$

(a) What are the boundary conditions for the electric field on either side of an interface between two materials? Given that for a single electromagnetic traveling wave $\vec{\mathbf{B}}(\vec{\mathbf{r}},t) = \frac{\vec{\mathbf{k}} \times \vec{\mathbf{E}}(\vec{\mathbf{r}},t)}{\omega}$ find the relationship between E_{0I} and B_{0I}, and also find the irradiance of this wave in terms of E_{0I}.

(b) Suppose this wave hits the surface of a perfect reflector (reflection coefficient r = -1) at normal incidence placed at z = 0. Write down equations similar to (1) for the electric and magnetic fields of the *reflected* wave in terms of E_{0I}, k, ω, and ensure that each field has the correct phase (i.e. pure cosine or pure sine or something in between) and direction.

(c) If instead, the surface of the conductor is at an arbitrary position $z = z_0$, what is form of the reflected wave due to the incident waves given in Eq. 1?

(d) Describe the resultant electric and magnetic fields for case c) in the region $z < z_0$ both qualitatively *and* algebraically. What is the irradiance in this region?

3. In the classical theory of the dispersion of light in a transparent, nonmagnetic, dielectric medium we assume that the light wave interacts with atomic electrons that are bound in harmonic oscillator potentials. In the simplest case the medium contains N electrons per unit volume with charge q and an effective resonance angular frequency ω_0, and damping constant γ.

a) By treating the *complex* displacement of an electron, $\tilde{x}(t)$, derive the steady-state response of an electron to a linearly polarized electromagnetic plane wave of electric field amplitude E_0 and angular frequency ω, represented by a complex electric field $\tilde{\mathbf{E}}(t) = \mathbf{E_0} e^{-i\omega t}$. Determine the phase relationship between the driving field and the displacement under the following three conditions: $\omega \ll \omega_0$; $\omega = \omega_0$; $\omega \gg \omega_0$.

b) For this medium, find the complex susceptibility, $\tilde{\chi}$, of the medium in terms of N, q, m, ε_0, ω, ω_0 and γ.

c) Now assume that the damping term is negligible, and therefore the susceptibility is real. Show that the group velocity, v_g, for wave packets traveling through this medium, composed of harmonic waves with wavevectors given by $k = \frac{n\omega}{c} = \frac{\omega}{c}\sqrt{1+\chi}$ is given by:

$$\frac{1}{v_g} = \frac{1}{cn}\left[n^2 + \frac{\omega^2 \omega_p^2}{\left(\omega_0^2 - \omega^2\right)^2}\right],$$

where $\omega_p = \frac{Nq^2}{m\varepsilon_0}$.

d) Given that the phase velocity, v_p, of a harmonic wave is given by c/n, show that $v_p v_g$ must always be less than c^2.

<u>Some useful trigonometric formulae</u>

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$
$$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$$
$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$
$$\sin(A) \pm \sin(B) = 2\sin\left[\tfrac{1}{2}(A \pm B)\right]\cos\left[\tfrac{1}{2}(A \mp B)\right]$$
$$\cos(A) + \cos(B) = 2\cos\left[\tfrac{1}{2}(A + B)\right]\cos\left[\tfrac{1}{2}(A - B)\right]$$
$$\cos(A) - \cos(B) = -2\sin\left[\tfrac{1}{2}(A + B)\right]\sin\left[\tfrac{1}{2}(A - B)\right]$$