Physics 420 – Electromagnetic Waves

Midterm: Thursday 16th December. 2 hours.

Answer <u>all</u> questions. Show your working! Sum of all marks = 80.

1. (i) Consider a long straight cylindrical wire of electrical conductivity σ and radius a carrying a uniform axial current of current density $\boldsymbol{J}$. Calculate the magnitude and direction of the Poynting vector at the surface of the wire.

[5]

(ii) Consider a thick slab made from a good conductor of conductivity σ, permeability μ (= $\mu_0\mu_r$) and permittivity ε (= $\varepsilon_0\varepsilon_r$) irradiated at normal incidence by a plane monochromatic electromagnetic wave traveling along the z-axis. If the electric field amplitude at the surface of the slab (at $z = 0$) is E_0, evaluate the Poynting vector within the slab averaged over one time period. You may assume $\sigma >> \omega\varepsilon$).

[9]

2. (i) What are the general boundary conditions (in terms of $\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{D}$, $\boldsymbol{H}$) at an interface between any two media?

[4]

(ii) A plane wave in vacuum is incident on a planar surface of a non-conducting, uncharged linear isotropic material with $\varepsilon_r \neq 1, \mu_r \neq 1$. The wave is incident at angle θ from the surface normal, the transmitted wave has angle φ from the surface normal, and the magnetic field of the incident wave is parallel to the surface. With the aid of a diagram give the boundary conditions on the electric field amplitudes of the incident, reflected and transmitted waves (E_{0I}, E_{0R} and E_{0T}) in terms of θ, φ, ε_r and μ_r.

[5]

(iii) What is the relation between θ and φ (in terms of ε_r and μ_r only)?

[2]

(iv) Derive the transmitted wave amplitude, E_{0T}, in terms of the incident wave amplitude and θ, φ, ε_r and μ_r.

[7]

3. (i) Describe what is meant by *the local conservation of charge*. Use Maxwell's equations to derive an equation that demonstrates this.

[4]

(ii) Give (don't derive) the analogous equation for the local conservation of *linear momentum*.

a) What is the Maxwell stress tensor?

b) Briefly explain how you would calculate the total force on all charges in a given volume using the Maxwell stress tensor.

[8]

(iii) A uniformly charged sphere (with radius R and total charge Q) is centred at the origin. By considering the semi-infinite volume bounded by the plane $z = 0$, calculate the total force on all charges in the "upper" hemisphere. You may find it helpful to determine which terms of the Maxwell stress tensor are important to calculate before evaluating the appropriate integrals.

[10]

4. (i) A plane monochromatic electromagnetic wave is traveling in *vacuo* with electric field and magnetic fields:

$$\begin{aligned}\mathbf{E_I}(z,t) &= \mathbf{E_{0I}}\cos(kz-\omega t)\\ \mathbf{B_I}(z,t) &= \mathbf{B_{0I}}\cos(kz-\omega t)\end{aligned} \qquad \text{(Eq. 1)}$$

(a) Find the Poynting vector and the irradiance of this wave in terms of E_{0I}.

[3]

(b) Suppose this wave hits the surface of a perfect conductor ($\sigma = \infty$) at normal incidence. Write down equations similar to (1) for the electric and magnetic fields of the *reflected* wave (with amplitudes E_{0R} and B_{0R}, respectively) in terms of k, w, and (in general) a phase constant, δ_R.

[2]

(c) If the surface of the conductor is at $z = z_0$, what is the value of this phase constant? What is δ_R when $z_0 = 0$?

[5]

(d) Describe the resultant electric and magnetic fields in the region $z < z_0$ both qualitatively *and* algebraically. What is the Poynting vector and irradiance in this region?

[8]

(ii) A "Mooney rhomb" is to be used to modify the polarization state of incident light through two total internal reflections as shown in Figure 1 on page 3.

(a) What value of the apex angle, θ, is required for the light to follow the geometrical path shown?

[4]

(b) What is the minimum value of the refractive index of the rhomb such that light entering from vacuum follows this path?

[4]

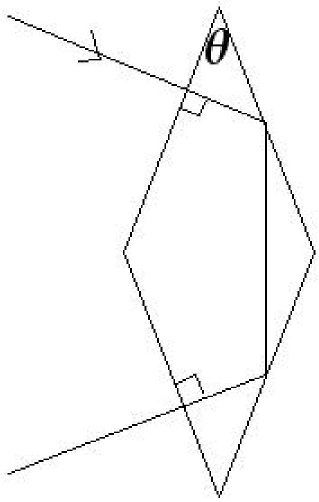

Fig 1.

Some useful formulae

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$
$$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$$
$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$
$$\sin(A) \pm \sin(B) = 2\sin\left[\tfrac{1}{2}(A \pm B)\right]\cos\left[\tfrac{1}{2}(A \mp B)\right]$$
$$\cos(A) + \cos(B) = 2\cos\left[\tfrac{1}{2}(A + B)\right]\cos\left[\tfrac{1}{2}(A - B)\right]$$
$$\cos(A) - \cos(B) = -2\sin\left[\tfrac{1}{2}(A + B)\right]\sin\left[\tfrac{1}{2}(A - B)\right]$$