Final Exam: Tuesday 20 April 2010

Allowed: **Time – three hours**
Formula sheets (given)
Calculator

Each question carries equal marks. Answer three out of the four questions.

For each problem briefly explain the equations you choose to use – simply writing down a set of equations from your formula sheet will not earn any marks.

Don't cram your answers into too small a space – try to spread out your answers.

1a) Describe what is meant by a *linear material*, and what this means for the $\vec{D}$ and $\vec{H}$ vector fields.
b) Describe what is meant by the term *skin depth*.
c) Derive the wave equation that is separately satisfied by the $\vec{E}$ and $\vec{B}$ fields within a linear conductor and, by considering an electromagnetic wave traveling along the *z*-axis, show that the skin depth in a highly-conducting material is given by

$$\delta \approx \sqrt{\frac{2}{\omega\sigma\mu}}$$

c) Silver is a good non-magnetic conductor, with conductivity given by $6.2\times10^{7}\ (\Omega\mathrm{m})^{-1}$, but it is expensive. If you wish to make a waveguide to operate at 10 GHz, how thick would you coat the walls?

2. Suppose we have a rectangular *wave guide* made from a good conductor, with one edge lying along the *z*-axis, and dimensions along the (x, y) directions of (a, b), where $a > b$.
a) What is meant by the term *TE mode* (here the reference direction is taken to be along $\hat{z}$)?
b) For a TE_{mn} mode the electric and magnetic fields inside the wave guide in terms of the amplitude of $\vec{B}$, i.e. B_0, are given by:

$$B_z(\vec{r},t) = B_0 \cos\left(\tfrac{m\,\pi\,x}{a}\right)\cos\left(\tfrac{n\,\pi\,y}{b}\right)\sin\left(kz - \omega\,t\right)$$

$$E_z(\vec{r},t) = 0$$

$$B_y(\vec{r},t) = \frac{k}{\left(\frac{\omega}{c}\right)^2 - k^2}\frac{n\pi}{b} B_0 \cos\left(\tfrac{m\,\pi\,x}{a}\right)\sin\left(\tfrac{n\,\pi\,y}{b}\right)\cos\left(kz - \omega\,t\right)$$

$$E_y(\vec{r},t) = \frac{\omega}{\left(\frac{\omega}{c}\right)^2 - k^2}\frac{m\pi}{a} B_0 \sin\left(\tfrac{m\,\pi\,x}{a}\right)\cos\left(\tfrac{n\,\pi\,y}{b}\right)\cos\left(kz - \omega\,t\right)$$

$$B_x(\vec{r},t) = \frac{-k}{\left(\frac{\omega}{c}\right)^2 - k^2}\frac{m\pi}{a} B_0 \sin\left(\tfrac{m\,\pi\,x}{a}\right)\cos\left(\tfrac{n\,\pi\,y}{b}\right)\cos\left(kz - \omega\,t\right)$$

$$E_x(\vec{r},t) = \frac{-\omega}{\left(\frac{\omega}{c}\right)^2 - k^2}\frac{n\pi}{b} B_0 \cos\left(\tfrac{m\,\pi\,x}{a}\right)\sin\left(\tfrac{n\,\pi\,y}{b}\right)\cos\left(kz - \omega\,t\right)$$

By picking one surface of the wave guide, show that these fields satisfy the expected boundary conditions.
d) In a length of the wave guide with $0 \le z \le 2\pi/k$ carefully deduce and sketch the form of both the electric and the magnetic fields for the TE_{10} mode at $t = 0$.

3. The Liénard-Wiechert retarded potentials for a moving point charge are given by:

$$V(\vec{r},t) = \frac{q}{4\pi\varepsilon_0 \wp} \frac{1}{1 - \frac{\hat{\wp}.\vec{v}}{c}}; \qquad \vec{A}(\vec{r},t) = \frac{\vec{v}}{c^2} V(\vec{r},t).$$

a) Explain each variable in these equations, and also, with the help of a clear diagram, what is meant by the term *retarded potential.*
b) A particle of charge q moves in a circle of radius a about the origin at constant angular velocity $\vec{\omega} = \omega\,\hat{z}$. At $t = 0$ the particle is at position $(a,0)$. Find the retarded potentials at any point on the z-axis at time t.

4. The *field tensor*, $F^{\mu\nu}$ is a *second-rank, antisymmetric, contravariant* tensor, and takes the form (with some elements missing – and denoted with "X"):

$$F^{\mu\nu} = \begin{pmatrix} X & E_x / c & E_y / c & E_z / c \\ X & X & B_z & -B_y \\ X & X & X & B_x \\ X & X & X & X \end{pmatrix}$$

a) Describe what is meant by the terms *second-rank, antisymmetric* and *contravariant* and complete the blanks (fill in the "*X*'s").
b) Write down the transformation rule for a general component of this tensor under a special Lorentz transformation between two inertial frames S and $\bar{S}$ with uniform motion along their common x-axes and origins overlapping at $t = \bar{t} = 0$.
c) The Lorentz transformation that gives the z-component of the electric field in frame $\bar{S}$ from the electromagnetic field quantities in frame S is:

$$\bar{E}_z = \gamma(v)\left(E_z + vB_y\right)$$

Confirm that this relationship is satisfied by the field tensor above and the rule you wrote down in (b).