Physics 421H – Electromagnetic Theory: 2005-2006

Final Exam

Dec 13th, 2005

Allowed: **Time – three hours**
Two sheets of handwritten notes
Formula sheet (given)
Calculator

Each question carries equal marks. Answer <u>three</u> questions from the four listed. Show your working.

1. A point charge q moves with constant velocity $\boldsymbol{v}$ in the $\hat{\mathbf{z}}$ direction so that at time t it is at point Q with coordinates $(0,0,vt)$. Position P has coordinates $(b,0,0)$. Find, at time t:
a) $\mathscr{r}$ ('r-script at the retarded time') as a function of v, t, b and c [Hint: It is easier to solve for $\mathscr{r}$ directly, instead of solving for t_r, the retarded time, and then substituting for $\mathscr{r}$].
b) the scalar potential $V(t)$ at point P.
c) the vector potential $\boldsymbol{A}(t)$ at point P.
d) State which of the three components of the electric field do not depend *explicitly* on the vector potential.

2. Consider the static magnetic field given in Cartesian coordinates by:

$$\mathbf{B} = B_0\left(x\,\hat{x} - y\,\hat{y}\right)/\,a$$

a) Show that this field satisfies the relevant pair of Maxwell's equations in free space.
b) Sketch the field lines and indicate where and in which direction currents should be put to approximate such a magnetic field.
c) If an observer is moving with velocity $\mathbf{v} = v\,\hat{\mathbf{z}}$ (note: along *an* axis, but not the $\hat{\mathbf{x}}$ axis) at some location (x,y), what is the electric field that this observer would experience?
d) What electric potential would such an observer measure relative to that at the origin?

3. In the classical theory of the dispersion of light in a transparent, nonmagnetic, dielectric medium one can assume that the light wave interacts with atomic electrons which are bound in harmonic oscillator potentials. In the simplest case the medium contains N electrons per unit volume with a resonance angular frequency ω_0 and damping constant γ.

a) By treating the *complex* displacement of an electron, $\tilde{x}(t)$, derive the steady-state response of an electron to a linearly polarized electromagnetic plane wave of electric field amplitude E_0 and angular frequency ω, represented by a complex electric field $\tilde{\mathbf{E}}(t) = \mathbf{E_0} e^{i\omega t}$. Determine the phase relationship between the driving field and the displacement under the following three conditions: $\omega << \omega_0$; $\omega = \omega_0$; $\omega >> \omega_0$.

b) For this medium, find the complex susceptibility of the medium in terms of N, q, m, ε_0, ω, ω_0 and γ.

c) Now assume that the damping term is negligible. Show that the group velocity, v_g, for waves travelling through this medium with wavevector given by $k = \frac{n\omega}{c} = \frac{\omega}{c}\sqrt{1+\chi}$ is given by:

$$\frac{1}{v_g} = \frac{1}{cn}\left[n^2 + \frac{\omega^2 \omega_p^2}{\left(\omega_0^2 - \omega^2\right)^2}\right],$$

where $\omega_p = \frac{Nq^2}{m\varepsilon_0}$.

d) Given that the phase velocity, v_p, of a monochromatic wave with angular frequency ω is given by c/n, show that $v_p v_g$ must always be less than c^2.

4) This question is composed of two parts with equal weighting:

Part I: Consider plane, monochromatic, polarized light travelling along the z-axis.

a) Write down mathematical expressions for the electric fields, $\mathbf{E}(\mathbf{r}, t)$, of oppositely circularly-polarized waves in terms of waves linearly polarized along $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$. Define one of them to be "right" and the other "left". Show how linearly polarized light can result from the superposition of these two waves.

b) Consider the propagation of an electromagnetic wave through a medium whose index of refraction depends upon the state of polarization, such that

$$n_{\pm} = \alpha \pm \eta$$

where α and η are real and positive, and the plus and minus signs refer to "right" and "left" circularly-polarized plane waves respectively. Show that a linearly-polarized plane wave incident on such a medium at $z = 0$ has its plane of polarization rotated as it travels through the medium and find the first value of z at which the plane of polarization is (i) linear again, and (ii) returns to the same plane of polarization at which it entered the medium.

Part II: Consider a charged spherical shell of radius R and uniform surface charge density σ_0.

c) Determine the Maxwell stress tensor over all space.

d) Use the Maxwell stress tensor to find the force exerted on the 'southern' ($z<0$) hemisphere due to the 'northern' hemisphere.