Physics 420 – Electromagnetic Theory: 2004-2005

Final Exam

April 19th, 2005

Allowed: **Time – three hours**
2 sides of 8½×11" paper containing your notes
Formula sheet (given)
Calculator

Each question carries equal marks. Answer <u>three</u> questions from the four listed.

1. a) The relativistic Lagrangian for a particle with charge q and position vector $\boldsymbol{r}$ interacting with an electromagnetic field is given by:

$$L = -mc^2\sqrt{1-\dot{r}^2/c^2} + q\dot{\boldsymbol{r}}\cdot\boldsymbol{A} - qV,$$

where $V(x,y,z,t)$ and $\boldsymbol{A}(x,y,z,t)$ are the usual scalar and vector potentials.

i) Consider only the x-component of $\boldsymbol{A}$ and find $\dfrac{dA_x}{dt}$ in terms of $\dfrac{\partial A_x}{\partial x}, \dfrac{\partial A_x}{\partial y}, \dfrac{\partial A_x}{\partial z}$ and $\dfrac{\partial A_x}{\partial t}$

ii) Show that Lagrange's equation $\dfrac{\partial L}{\partial x} - \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{x}} = 0$ gives the x-component of the Lorentz force equation.

b) Consider plane, monochromatic, polarized light travelling along the z-axis.

i) Explain how an arbitrary state of polarized light can be created by the addition of components that are linearly-polarized along the x and y axes.

ii) Write down expressions for the electric fields of oppositely circularly-polarized plane waves and define one of them to be "right" and the other "left".

iii) Consider the propagation of an electromagnetic wave through a medium whose index of refraction depends upon the state of polarization, such that

$$n_{\pm} = \alpha \pm \eta$$

where α and η are real and positive, and the plus and minus signs refer to "right" and "left" circularly-polarized plane waves respectively. Show that a linearly-polarized plane wave incident on such a medium at $z = 0$ has its plane of polarization rotated as it travels through the medium and find the first value of z at which the plane of polarization returns to that at which it entered the medium.

2. A hollow box with perfectly conducting walls has inside dimensions a = 2 cm, b = 3 cm and d = 1 cm (in the x, y and z directions, respectively).
a) Use Maxwell's equations to deduce the boundary conditions for the electromagnetic field inside this box.
b) Write down the general form of the electric and magnetic fields for the mnl^{th} mode inside this box, and derive an expression for the angular frequency of this mode.
c) If we define the wavelength, λ, by $\lambda = \frac{2\pi c}{\omega}$, how many modes of wavelength λ in the range $4/\sqrt{5} \le \lambda \le 8/\sqrt{13}$ cm are there? What are they, and what are their wavelengths?
d) Make a rough (order-of-magnitude) estimate of how many modes are in the range $0.01 \le \lambda \le 0.011$ cm.

3. a) Starting with the four Maxwell's equations, derive Maxwell's equations in potential form in an arbitrary gauge.
b) Use the Lorentz gauge to simplify these equations. Write down the solutions $V(\mathbf{r},t)$ and $\boldsymbol{A}(\mathbf{r},t)$ for the **specific** case of a *point* charge. What name do we give to these potentials? Explain fully each term present and, where appropriate, what it physically represents.
c) A point charge of charge q moves in a circle about the origin at constant angular velocity ω in the xy plane (with this vector lying along $+\hat{z}$). At t = 0 the particle is at position (a ,a). Find the scalar and vector potentials at any point on the z-axis at time t.
d) Can you derive the electric and magnetic fields from this result?

4. Two observers S and S' (with S' travelling at speed v with respect to S along their common $\hat{x}$ -direction) each measure the linear momentum of a particle, the wavevector of a photon and the electric and magnetic fields in their laboratories. Suppose that S measures a linear momentum of $p^{\mu} = (E/c, p, 0, 0)$, a (4-vector) wavevector of $k^{\mu} = (\omega/c, k^x, k^y, 0)$ and a field strength tensor of:

$$F^{\mu\upsilon} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & -B & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

a) Define what is meant by a *4-vector* and by a *field strength tensor*.
b) Determine what S' measures for the linear momentum of the particle, the wavevector of the photon and the field strength tensor.
c) Calculate $p^{\mu}k_{\mu}$ and $F^{\mu\upsilon}F_{\mu\upsilon}$ in each frame. What invariants have you discovered?