Midterm: Wednesday Nov 4, 2026

Allowed: Formula sheet (given), calculator, 1 hour 50 minutes

All questions carry equal marks

1. A particle that can lie anywhere along the *x*-axis is described by the wave function:

$$\Psi(x,t) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} e^{-\frac{i\omega t}{2}}$$

Show that this wave function is correctly normalized.

2. Consider the quantum mechanical behaviour of a "free particle" of mass *m*.

(a) What is meant by the term *free particle*? Write down the 1-D Schrödinger equation for this case. Give an example of a particle that is *not* free.

(b) Show that the trial function $\Psi(x,t) = Ae^{i(kx-\omega t)}$ is a solution of the Schrödinger equation provided a particular relationship between ω and k is satisfied. What is this relationship? Sketch this with ω on the vertical axis and k on the horizontal axis.

(c) If a particle's wave function is approximately given by $\Psi(x,t) \sim A\exp\left(i(9.12\times10^{11}x - 2.64\times10^{16}t)\right)$, with all values and variables in SI units, then determine the particle's momentum, total energy, and mass.

3. A particle of mass *m* in an infinite square well with walls at $x = 0$ and $x = a$ has the initial wave function

$$\Psi(x,0) = \begin{cases} Ax & 0 \le x \le a/2 \\ A(a-x) & a/2 \le x \le a \end{cases}$$

(a) Sketch $\Psi(x,0)$ and $|\Psi(x,0)|^2$, and determine the constant *A*.

(b) This wave function can be written as

$$\Psi(x,0) = \sum_{n=1,3,5\ldots} (-1)^{\frac{n-1}{2}} \frac{4\sqrt{6}}{n^2\pi^2}\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a} .$$

(i) Find the probability that a measurement of the energy would yield the ground state energy, and also the energy of the $n = 3$ state.

(ii) Write down the expression for the time-dependent wave function $\Psi(x,t)$ in terms of *m*, *a*, and $\hbar$.

continued overleaf…

4. A particle at $t = 0$ is described by the following wave function:

$$\Psi(x,0) = \sqrt{\frac{1}{a}} \qquad |x| < \frac{a}{2}$$

$$\Psi(x,0) = 0 \qquad |x| > \frac{a}{2}$$

By expanding this function as an integral over the momentum eigenfunctions:

$$\Psi(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k)\, e^{ikx}\, dk\,,$$

determine $\phi(k)$ and find the probability that a measurement of momentum (where $p = \hbar k$) would yield a value between $-2\pi\hbar/a$ and $2\pi\hbar/a$.

[You may find one of the following integrals useful:

$\int_{-\pi}^{\pi} \frac{\sin^2 x}{x^2} dx = 0.903\pi$ $\qquad$ $\int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{x^2} dx = 0.774\pi$]

5. The *recursion relation* that results from an analytical approach to finding the *n*-th energy eigenfunction ψ_n for a particle of mass *m* in a harmonic potential (with $V(x) = \frac{1}{2} m\omega^2 x^2$) is

$$a_{j+2} = \frac{-2(n-j)}{(j+1)(j+2)} a_j\,,$$

where for each energy eigenfunction, $\psi_n(\xi) = A\, h(\xi)\, e^{-\xi^2/2}$, $h(\xi) = \sum_0^{\infty} a_j \xi^j$, and $\xi = \sqrt{\frac{m\omega}{\hbar}}\, x$.

(a) Given that the energy eigenfunctions for this 1-D case *must be* either even or odd, find the expression for $\psi_6(x)$ explicitly in terms of *x* (retain both *A* and a_0 in your expression).

(b) If $\psi_6(x)$ is normalized when the highest power of *x* has the numerical pre-factor $\frac{8}{\sqrt{6!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \left(\frac{m\omega}{\hbar}\right)^3$, find the magnitude of the product $A\, a_0$.