Midterm: 2011FA

Allowed: Formula sheets (given), calculator, 3 hours

Do <u>all</u> of question 1, and <u>two</u> out of <u>three</u> of the remaining questions.
All questions are worth equal marks. Show your working!

PART A: Answer <u>all</u> parts of this question

1.a) Show that $e^{-x^2/2}$ is an eigenfunction of the operator $\hat{Q} = \frac{d^2}{dx^2} - x^2$ and find its eigenvalue.

b) Show, using the defining properties of Dirac delta functions and an arbitrary physically-realistic function $f(x)$, that $x\delta'(x) = -\delta(x)$. Pick a function of your choice that in the limiting case approaches a Dirac delta function, and sketch for a range of *x*-values except $x = 0$ both sides of this equation.

c) Define what is meant by an *Hermitian operator*, and show both that the eigenvalues of a Hermitian operator are necessarily real and all non-degenerate eigenfunctions are necessarily orthogonal.

d) At some instant of time a free particle in 1-D has the following state function:

$$\psi(x) = Ae^{ikx} + \frac{A}{\sqrt{3}}e^{-ikx}$$

In principle what values of momentum will be measured, and with what relative probabilities? Explain your reasoning.

e) At some instant of time an electron in a one-dimensional box with walls at $x = (0, a)$ has the following state function:

$$\psi(x) = \begin{cases} A & 0 < x < a/2 \\ -A & a/2 < x < a \end{cases}$$

Find the normalization constant, *A*, and the lowest energy that can be measured in this state.

f) Show that J_x for a 1-D wave function of the form $\Psi(x,t) = Ae^{i\phi(x,t)}$ is given by $J_x = \frac{\hbar}{m}|A|^2 \frac{\partial \phi}{\partial x}$.

g) By expanding the exponential of a matrix in terms of its Taylor expansion, and using the binomial expansion from your formula sheet, show that only if two matrices, $\underline{\underline{M}}$ and $\underline{\underline{N}}$, commute, then $\exp(\underline{\underline{M}} + \underline{\underline{N}}) = \exp(\underline{\underline{M}})\exp(\underline{\underline{N}})$.

PART B: Answer <u>two</u> out of the following <u>three</u> questions

2. At time $t = 0$ a particle is represented by the state function

$$\Psi(x,0) = \begin{cases} Ax/a & \text{if } 0 \le x \le a \\ A(b-x)/(b-a) & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

a) Normalize Ψ to show that $A = \sqrt{3/b}$

b) Sketch $\Psi(x,0)$ as a function of *x*.

c) In which region of space is the particle most likely to be found at $t = 0$?

d) What is the probability of finding the particle to the left of *a*?

e) What is the expectation value of *x*?

3. Consider the following equation on your formula sheet:

$$\frac{d}{dt}\langle\hat{O}\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{O}]\rangle + \left\langle\frac{\partial}{\partial t}\hat{O}\right\rangle$$

a) Derive this equation by expanding and manipulating the left-hand side, then employing the time-dependent Schrödinger equation.

b) Using this equation, and considering the Hamiltonian for a single particle experiencing a potential energy $V(x)$, derive **both** of the two equations below:

$$m\frac{d\langle x\rangle}{dt} = \langle p\rangle \qquad \underline{\text{and}} \qquad \frac{d\langle p\rangle}{dt} = \left\langle -\frac{\partial V}{\partial x}\right\rangle$$

What classical laws/equations do they correspond to?

4. Consider a 2-dimensional Hilbert space with two basis states

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad ; \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and the Hamiltonian is represented in this basis by the following matrix:

$$\underline{\underline{H}} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

where *a* and *b* are real constants.

i) Find the eigenvalues and normalized eigenvectors of this Hamiltonian. Note that the energy eigenstates are now described in terms of the basis $\{|1\rangle, |2\rangle\}$.

ii) Suppose the system starts out in state $\{|1\rangle\}$. Show that the state at time *t* is given by:

$$|\Psi(t)\rangle = e^{-iat/\hbar}\begin{pmatrix} \cos(bt/\hbar) \\ -i\sin(bt/\hbar) \end{pmatrix}$$