Final Exam: Sunday Apr 16, 2023 Name:____________________

Allowed: Formula sheet (given), calculator, 2½ hours

PART 1 – answer questions **1-7** in your exam booklet and **then** use your scratch card:
One scratch = **100%**; two scratches **50%**; three scratches **33%**; four scratches **25%** (part 1 total is 40%)
PART 2 – answer questions **8–10** in the exam booklet provided (each question here is worth 20%)

Qu's 1 - 3) Consider a beam of particles of mass m and energy E travelling from $x<0$, where $U=0$, towards a potential energy step with height U_0 located at $x=0$, and with $E>U_0$.
Harris examines this problem, and adopts a form for the wave function for each particle in each region given by

$$\psi_{x<0}(x) = Ae^{ikx} + Be^{-ikx} \quad \text{and} \quad \psi_{x>0}(x) = Ce^{ik'x}$$

The following relationships are then found to hold between A, B and C:

$$B = \frac{k-k'}{k+k'} \quad \text{and} \quad C = \frac{2k}{k+k'}$$

1) Which of the following are the correct expressions for k and k'?

A. $k=\sqrt{\dfrac{2mE}{\hbar^2}}$; $k'=\sqrt{\dfrac{2m(E-U_0)}{\hbar^2}}$

B. $k=\sqrt{\dfrac{2mE}{\hbar^2}}$; $k'=\sqrt{\dfrac{2m(U_0-E)}{\hbar^2}}$

C. $k=\sqrt{\dfrac{2m(E-U_0)}{\hbar^2}}$; $k'=\sqrt{\dfrac{2m(U_0-E)}{\hbar^2}}$

D. $k=\sqrt{\dfrac{2m(E+U_0)}{\hbar^2}}$; $k'=\sqrt{\dfrac{2m(E-U_0)}{\hbar^2}}$

E. $k=\sqrt{\dfrac{2mE}{\hbar^2}}$; $k'=\sqrt{\dfrac{2m(E+U_0)}{\hbar^2}}$

2) Suppose that $U_0=-3E$ (and so this is in fact a potential energy *drop*, for which Harris' equations still hold), and for ease we shall let A adopt the value 1. Which of the following are then viable forms of the wavefunction in each region?

A. $\psi_{x<0}(x)=e^{ikx}-\frac{1}{3}e^{-ikx}$, $\psi_{x>0}(x)=\frac{1}{3}e^{ik'x}$

B. $\psi_{x<0}(x)=e^{ikx}+\frac{1}{3}e^{-ikx}$, $\psi_{x>0}(x)=\frac{2}{3}e^{ik'x}$

C. $\psi_{x<0}(x)=e^{ikx}-\frac{1}{3}e^{-ikx}$, $\psi_{x>0}(x)=\frac{2}{3}e^{ik'x}$

D. $\psi_{x<0}(x)=e^{ikx}-\frac{2}{3}e^{-ikx}$, $\psi_{x>0}(x)=\frac{2}{3}e^{ik'x}$

E. $\psi_{x<0}(x)=e^{ikx}+\frac{2}{3}e^{-ikx}$, $\psi_{x>0}(x)=\frac{2}{3}e^{ik'x}$

3) What fraction of incident particles reflect from this potential energy drop?

A. $R=0.45$ **B.** $R=0.25$ **C.** $R=0.11$ **D.** $R=0.33$ **E.** $R=0.66$

4) The one-particle orbital angular momentum operator, $\hat{\mathbf{L}}$, is defined in terms of its constituent operators: $\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$. In three dimensions how are $\hat{\mathbf{r}}$ and $\hat{\mathbf{p}}$ defined?

A. $\hat{\mathbf{r}}$ is x and $\hat{\mathbf{p}}$ is $-i\frac{\partial}{\partial x}$

B. $\hat{\mathbf{r}}$ is the vector (x, y) and $\hat{\mathbf{p}}$ is the vector $-i\hbar\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

C. $\hat{\mathbf{r}}$ is the vector (x, y) and $\hat{\mathbf{p}}$ is the vector $i\hbar\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

D. $\hat{\mathbf{r}}$ is the vector (x, y, z) and $\hat{\mathbf{p}}$ is the vector $i\hbar\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

E. $\hat{\mathbf{r}}$ is the vector (x, y, z) and $\hat{\mathbf{p}}$ is the vector $-i\hbar\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

5) If an electron is in a p state, what are the possible values of L_z, in units of $\hbar$?

A. $-3, -2, -1, 0, +1, +2, +3$ **B.** $-1, 0, +1$ **C.** $-2, -1, 0$ **D.** $-2, -1, 0, +1, +2$ **E.** $0, +1, +2$

6) For an electron in a p state what is the *smallest* possible angle between its orbital angular momentum vector and the *z*-axis?

A. 45° **B.** 30° **C.** 23° **D.** 18° **E.** 12°

7) Consider two different possible wave functions for two electrons in an infinite well, with each wave function composed of (different) spatial and spin parts:

$$\psi_a(1,2) = \phi_a(\vec{r}_1, \vec{r}_2)\uparrow_1\uparrow_2 \qquad \psi_b(1,2) = \phi_b(\vec{r}_1, \vec{r}_2)\frac{1}{\sqrt{2}}(\uparrow_1\downarrow_2 - \downarrow_1\uparrow_2)$$

Which of the following statements must be true about the *exchange symmetry* of the spatial and spin parts of these wave functions?

A. the spin part of ψ_a is antisymmetric and the spatial part of ψ_a is symmetric

B. the spin part of ψ_b is antisymmetric and the spatial part of ψ_b is symmetric

C. the spin part of ψ_a is symmetric and the spatial part of ψ_b is asymmetric (has no symmetry)

D. the spatial part of ψ_a is symmetric and the spin part of ψ_b is antisymmetric

E. the spin part of ψ_a is symmetric and the spatial part of ψ_b is antisymmetric

PART II – answer all <u>three</u> questions in exam booklet provided

8. A particle of mass m experiences a potential energy given by $\frac{1}{2}\kappa x^2$. It occupies the first excited energy state, with wavefunction:

$$\psi(x) = Axe^{-\left(\frac{m\kappa}{\hbar^2}\right)^{\frac{1}{2}}\frac{x^2}{2}}$$

where A is a normalization constant.

a) Evaluate the expectation value of x and the expectation value of p in this state. (Hint: Use symmetry if possible and explain why this is of help).

b) Find, but do not calculate, the integral (written in terms of x) that gives the expectation value of p^2 in this state.

9. Consider the 3d electron in a hydrogen atom.

a) Carefully sketch, with reasoning and labelled axes with units, the *radial probability density*, $P(r)$, for this electron. Demonstrate mathematically that the peak of this function occurs at $r = 9\,\mathrm{a}_0$, and explain the significance of this.

b) Estimate the probability that the electron may be found within a total range of distance of $\Delta r = 0.03\,\mathrm{a}_0$ in the region of $r \approx 9\,\mathrm{a}_0$.

10. Two identical particles are in an infinite well and together they occupy both the ground state and the first excited state.

a) Write down the *spatial* wave functions of these two particles that are: (i) symmetric, and (ii) antisymmetric with respect to particle exchange (neglect any spin part of this wavefunction), taking the normalization constant of these wave functions to be *A*.

b) Write down but do not calculate the integral (which is composed of three added/subtracted parts), that gives the probability of one particle lying between $x_a \to x_b$ and the other particle lying between $x_c \to x_d$. If these two regions overlap substantially which wave function from part (a) will give the greater probability of finding the particles in these two regions, and why?

[Note that the spatial wave function for one particle in the *n*-th state of an infinite well is $\psi_n(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}$]